Discussions on a mathematical induction problem

Question A sequence of real numbers $\{a_n\}$ is defined as follows.

$a_0 = a_1 = 1, \qquad a_2 = 3$ and

$a_{n+3} = 3a_{n+2} - a_{n+1} - 2a_n \quad , \qquad n = 0, 1, 2, \ldots.$

(a) Let $b_k = a_{k+2} - a_{k+1} - a_k \quad , \qquad k = 0, 1, 2, \ldots.$

Prove that $b_n = 2b_{n-1}$ for all $n \in \mathbb{N}$.

(b) Hence, or otherwise, deduce that $a_n \le 2^n$.

"Proposed" solution

(a) Let P(n) be the proposition : " $b_n = 2b_{n-1}$ ".

For P(1), $b_0 = a_2 - a_1 - a_0 = 3 - 1 - 1 = 1$

$b_1 = a_3 - a_2 - a_1 = (3a_2 - a_1 - 2a_0) - a_2 - a_1 = (3 \times 3 - 1 - 2 \times 1) - 3 - 1 = 2$

$\therefore \quad b_1 = 2b_0$.

Assume P(k) is true for some $k \in \mathbb{N}$, that is , $b_k = 2b_{k-1}$ …. (1)

For P(k + 1),

$$\begin{aligned} b_{k+1} &= a_{k+3} - a_{k+2} - a_{k+1} \\ &= 3a_{k+2} - a_{k+1} - 2a_k - a_{k+2} - a_{k+1} \\ &= 2(a_{k+2} - a_{k+1} - a_k) \\ &= 2b_k \end{aligned}$$

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall\, n \in \mathbb{N}$.

(b) Let P(n) be the proposition : " $a_n \le 2^n$ ".

For P(0), $a_0 = 1 \le 2^0$

For P(1), $a_1 = 1 \le 2^1$

For P(2), $a_2 = 1 \le 2^2$ $\therefore$ P(0) , P(1), P(2) are true.

Assume P(k), P(k + 1) and P(k + 2) are true for some $k \in \mathbb{N}$, that is,

$a_k \le 2^k, \qquad a_{k+1} \le 2^{k+1}, \qquad a_{k+2} \le 2^{k+2}$ …. (2)

For P(k + 3) ,

$$\begin{aligned} a_{k+3} &= 3a_{k+2} - a_{k+1} - 2a_k && \text{(Note: we cannot use (2) directly as there are subtractions)} \\ &= 3(a_{k+2} - a_{k+1} - a_k) + 2a_{k+1} + a_k \\ &= 3b_k + 2a_{k+1} + a_k \\ &= 3(2b_{k-1}) + 2a_{k+1} + a_k && \text{, by (a)} \\ &= 3(4b_{k-2}) + 2a_{k+1} + a_k && \text{, by (a)} \\ &= \ldots. \\ &= 3(2^k b_0) + 2a_{k+1} + a_k \\ &\le 3(2^k \times 1) + 2(2^{k+1}) + 2^k \\ &= 2^k[3 + 2(2) + 1] \\ &= 2^{k+3} && \therefore \quad \text{P(k + 3) is true.} \end{aligned}$$

By the Second Principle of Mathematical Induction, P(n) is true $\forall\, n \in \mathbb{N} \cup \{0\}$.

Discussions

In the above "Proposed" solution,

(1) The mathematical induction in **(a)** is wrong, as the inductive hypothesis ($b_k = 2b_{k-1}$) is not used in proving $P(k+1)$ part. Therefore either you forfeit the use of mathematical induction and prove directly as in Method 1 below or modify your mathematical induction as in Method 2 below.

Method 1

$$\begin{aligned} b_n &= a_{n+2} - a_{n+1} - a_n \\ &= (3a_{n+1} - a_n - 2a_{n-1}) - a_{n+1} - a_n \\ &= 2(a_{n+1} - a_n - a_{n-1}) \\ &= 2b_{n-1} \end{aligned}$$

Method 2

Let $P(n)$ be the proposition : " $b_n = 2b_{n-1}$ ".

For $P(1)$, $b_0 = a_2 - a_1 - a_0 = 3 - 1 - 1 = 1$

$b_1 = a_3 - a_2 - a_1 = (3a_2 - a_1 - 2a_0) - a_2 - a_1 = (3\times 3 - 1 - 2\times 1) - 3 - 1 = 2$

$\therefore$ $b_1 = 2b_0$.

Assume $P(k)$ is true for some $k \in \mathbb{N}$, that is , $b_k = 2b_{k-1}$ …. (1)

For $P(k+1)$,

By (1),

$$\begin{aligned} b_k &= 2b_{k-1} \\ a_{k+2} - a_{k+1} - a_k &= 2(a_{k+1} - a_k - a_{k-1}) \\ 2a_{k+2} - 2a_{k+1} - 2a_k &= 2(2a_{k+1} - 2a_k - 3a_{k-1}) \\ (3a_{k+2} - a_{k+1} - 2a_k) - a_{k+2} - a_{k+1} &= 2[(3a_{k+1} - a_k - 2a_{k-1}) - a_{k+1} - a_k] \\ 2(a_{k+3} - a_{k+2} - a_{k+1}) &= 2[2(a_{k+2} - a_{k+1} - a_k)] \\ a_{k+3} - a_{k+2} - a_{k+1} &= 2[a_{k+2} - a_{k+1} - a_k] \\ b_{k+1} &= 2b_k \end{aligned}$$

$\therefore$ $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true $\forall\, n \in \mathbb{N}$.

As can be seen, Method 1 is better than Method 2.

(2) The Proposed solution in Part **(b)** is done better than part **(a)** but is not satisfactory:

(i) The deduction in the $P(k+3)$ part as denoted by " = …. " is not wrong , but rather uncomfortable. You may better write " = …. (use deduction, hence induction)" to tell others that you know how to use induction here, but you are not writing for simplicity.

(ii) $P(k+2)$, that is , $a_{k+2} \le 2^{k+2}$ is not really used in the proof of $P(k+3)$ in the above. You may cut that part, but it is still all right to leave it there.

You can in fact prove separately $b_n = 2^n$ by mathematical induction (or deduction) first and the proof of part (b) can be rewritten as follows:

Let P(n) be the proposition : " $a_n \leq 2^n$ " .

For P(0), $a_0 = 1 \leq 2^0$

For P(1), $a_1 = 1 \leq 2^1$

For P(2), $a_2 = 1 \leq 2^2$ $\therefore$ P(0) , P(1), P(2) are true.

Assume P(k) and P(k + 1) are true for some $k \in \mathbb{N}$, that is,

$$a_k \leq 2^k, \qquad a_{k+1} \leq 2^{k+1} \qquad \text{…. (2)}$$

For P(k + 3) ,

$$\begin{aligned} a_{k+3} &= 3a_{k+2} - a_{k+1} - 2a_k \\ &= 3(a_{k+2} - a_{k+1} - a_k) + 2a_{k+1} + a_k \\ &= 3b_k + 2a_{k+1} + a_k \\ &\leq 3 \times 2^k + 2 \times 2^{k+1} + 2^k \\ &= 3 \times 2^k + 4 \times 2^k + 2^k \\ &= 8 \times 2^k \\ &= 2^{k+3} \end{aligned}$$

$\therefore$ P(k + 3) is true.

By the Second Principle of Mathematical Induction, P(n) is true $\forall\, n \in \mathbb{N} \cup \{0\}$.